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Core-Plus Mathematics
Using the Unit Resource Masters

Overview of Unit Resource Masters

To assist you as you teach Course 1 of Core-Plus Mathematics, this unit-specific resource book has been developed. The unit resources provided can help you focus student attention on the important mathematics being developed. They can be used to help students organize their results related to specific problems, synthesize what they are learning, and practice for standardized tests.

Each unit resource book provides the following masters in the order that they are used in the unit.

- **Transparency Masters**
  1. Think About This Situation (TATS) masters to help launch the lesson
  2. Masters to collect class results
  3. Summarize the Mathematics (STM) masters to help facilitate the synthesis of mathematical ideas from the investigation (To guide your planning, sample discussion scenarios called “Promoting Mathematical Discourse” are provided in the Teacher’s Guide for selected TATS and STM discussions.)

- **Student Masters**
  1. Masters to help students organize their results
  2. Technology Tips to facilitate learning technology feature of graphing calculators, spreadsheet software, and computer algebra systems (CAS)
  3. Unit Summary masters to provide a starting point for pulling together the main mathematical ideas of a unit
  4. Practicing for Standardized Tests masters provide an opportunity for students to complete tasks presented in the format of most high-stakes tests and to consider test-taking strategies. (Solutions to these tasks are printed in the Teacher’s Guide following the final unit Summarize the Mathematics. This allows you the option of providing or not providing the solutions to students.)

- **Assessment Masters**
  1. Quizzes (two forms for each lesson)
  2. In-class tests (two forms for each unit)
  3. Take-home assessment items (three items for each unit)
  4. Projects (two or three for each unit)
  5. Midterm and end-of-course assessment items (Unit 4 and Unit 8 contain a bank of assessment items from which to design cumulative exams.)

All of the items in this book are included for viewing and printing from the Core-Plus Mathematics TeacherWorks Plus CD-ROM. Custom tailoring of assessment items in this book, as well as creation of additional items, can be accomplished by using the ExamView Assessment Suite.
Throughout the *Core-Plus Mathematics* curriculum, the term “assessment” is meant to include all instances of gathering information about students’ levels of understanding and their disposition toward mathematics for purposes of making decisions about instruction. The dimensions of student performance that are assessed in this curriculum (see chart below) are consistent with the assessment recommendations of the National Council of Teachers of Mathematics in the *Assessment Standards for School Mathematics* (NCTM, 1995). They are more comprehensive than those of a typical testing program.

<table>
<thead>
<tr>
<th>Assessment Dimensions</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Process</strong></td>
</tr>
<tr>
<td>Problem Solving</td>
</tr>
<tr>
<td>Reasoning</td>
</tr>
<tr>
<td>Communication</td>
</tr>
<tr>
<td>Connections</td>
</tr>
</tbody>
</table>

These unit resource masters contain the tools for formal assessment of the process and content dimensions of student performance. Calculators are assumed in most cases on these assessments. Teacher discretion should be used regarding student access to their textbooks and Math Toolkits for assessments. In general, if the goals to be assessed are problem solving and reasoning, while memory of facts and procedural skill are of less interest, resources should be allowed. However, if automaticity of procedures or unaided recall are being assessed, it is appropriate to prohibit resource materials.

You may want to consult the extended section on assessment in the front matter of the Course 1 *Core-Plus Mathematics Teacher’s Guide* and *Implementing Core-Plus Mathematics*. Among the topics presented in these sources are curriculum-embedded assessment, student-generated assessment, and scoring assessments and assigning grades. Since the *Core-Plus Mathematics* approach and materials provide a wide variety of assessment information, the teacher will be in a good position to assign grades. With such a wide choice of assessment opportunities, a word of caution is appropriate: *It is easy to overassess students, and care must be taken to avoid doing so.* Since many rich opportunities for assessing students are embedded in the curriculum itself, you may choose not to use a quiz at the end of every lesson or to replace all or portions of an in-class test with take-home tasks or projects.
Think About This Situation

Suppose that you and two or three friends each grabbed the rope at a different knot and pulled outward until the loop formed a particular shape.

a How could you position yourselves so that the resulting shape was an equilateral triangle? An isosceles triangle? A right triangle?

b How are the perimeters of the three triangles related? How do you think the areas are related?

c How could you position yourselves so that the resulting shape was a square? A rectangle? A parallelogram that is not a rectangle?

d How are the perimeters of the three quadrilaterals related? How do you think the areas are related?
In this investigation, you experimented with building triangles and quadrilaterals with different side lengths. You also investigated how the rigidity of triangles and the nonrigidity of quadrilaterals influence their uses in the design of structures and devices.

a Describe the similarities and differences in what you discovered in your triangle-building and quadrilateral-building experiments.

b Suppose you are told that a triangular garden plot is to have sides of length 5 m, 12 m, and 13 m.

   i. Explain why it is possible to have a triangular plot with these dimensions.

   ii. Explain how you and a partner could lay out such a plot using only a 15-meter tape measure.

   iii. How many differently shaped triangular plots could be laid out with these dimensions? Why?

c What constraints are needed on the lengths of the sides of a quadrilateral for it to be a parallelogram? What additional constraint(s) are needed for it to be a rectangle?

d What does it mean to say that a shape is rigid? How can you make a quadrilateral rigid?

e What must be true about the sides of a quadrilateral linkage if one of the cranks can make a complete revolution? If both cranks can make complete revolutions?

Be prepared to share your ideas and reasoning with the class.
In this investigation, you discovered combinations of side lengths or angle measures that were sufficient to determine if two triangles were congruent. You also explored how you could use congruent triangles to reason about properties of an isosceles triangle.

**a** Which sets of conditions—SSS, SAS, SSA, ASA, and AAA—can be used to test if two triangles are congruent?

**b** Write each Triangle Congruence Condition in words and illustrate with a diagram.

**c** If $\triangle PQR \cong \triangle XYZ$, what segments are congruent? What angles are congruent?

**d** Describe properties of an isosceles triangle that you know by definition or by reasoning.

*Be prepared to share your ideas and reasoning with the class.*
Exploring SSA Triangle Congruency
Check Your Understanding Extension

How does this diagram illustrate that SSA does not determine congruency of triangles?
In this investigation, you used Triangle Congruence Conditions to support your reasoning about properties of shapes.

a What is true about any point on the perpendicular bisector of a segment? How is this related to congruence of triangles?

b What is the sum of the measures of the interior angles of any quadrilateral? How could you convince others of this property?

c What are some special properties of parallelograms? Of rectangles? How are these properties related to congruence of triangles?

d What are some general strategies to consider when trying to establish properties of shapes by reasoning?

*Be prepared to share your ideas with the class.*
The Pythagorean Theorem

The diagram illustrates the Pythagorean Theorem, which states that in a right-angled triangle, the square of the length of the hypotenuse (c) is equal to the sum of the squares of the other two sides (a and b). Mathematically, this is expressed as:

\[ a^2 + b^2 = c^2 \]
In this investigation, you examined applications of the Pythagorean Theorem and its converse. You also used careful reasoning to provide arguments for why these statements are true.

a Describe the general idea behind your argument that the Pythagorean Theorem is true for all right triangles.

b Describe the general idea behind your argument that the converse of the Pythagorean Theorem is true.

c Give two examples, one mathematical and one not involving mathematics, to illustrate that if a statement is true, its converse may not be true.

d What is the smallest number of side lengths you need to compare in order to test if two right triangles are congruent? Does it make a difference which side lengths you use? Explain.

Be prepared to share ideas and examples with the class.
1. In this lesson, you learned that AAA and SSA conditions for triangles do not always determine exactly one triangle. Pick one of these conditions and explain to someone not in this class why it does not always determine exactly one triangle.

2. You have a rope that is 15 meters long with a knot at every meter.
   a. Explain how three people could hold the rope so that an isosceles triangle is formed.
   b. Three people hold the rope so that the lengths of the sides of the triangle that is formed are 4 m, 5 m, and 6 m. Is the triangle a right triangle? Explain your reasoning.
3. Quadrilateral $ABCD$ is a rhombus.

![Diagram of a rhombus]

a. Must diagonals $DB$ and $AC$ be the same length? Explain your reasoning.

b. Does $DB$ bisect $\angle D$? Provide reasoning to support your answer.

4. In the building truss below, board $BD$ is positioned so that if it were extended, it would be the perpendicular bisector of board $AC$.

![Diagram of a building truss]

Explain why boards $AB$ and $CB$ should be cut the same length.
1. The AAA condition does not always determine exactly one triangle because you could have triangles that are the same shape but different sizes. For example, any two equilateral triangles have all angles with measure 60°, but they will not be the same triangle unless the side lengths are the same also.

The SSA condition often allows one to build two different triangles by changing the measure of the angle between the two given sides. For example, if you have sides of length 5 and 10 and an angle measure of 20°, you can create two different triangles \( \triangle ACB \) and \( \triangle DCB \) as shown at the right.

2. a. All possibilities with integer side lengths are provided in the table below.

<table>
<thead>
<tr>
<th>Side Length 1</th>
<th>Side Length 2</th>
<th>Side Length 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>4</td>
<td>7</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>3</td>
</tr>
<tr>
<td>7</td>
<td>7</td>
<td>1</td>
</tr>
</tbody>
</table>

b. The triangle is not a right triangle because \( 4^2 + 5^2 = 16 + 25 = 41 \neq 6^2 \).

3. a. Diagonals \( \overline{DB} \) and \( \overline{AC} \) do not need to be the same length. Since quadrilaterals aren’t rigid, the rhombus could have a shape like \( \diamond \). Clearly the diagonals in this rhombus are not the same length.

b. Yes, \( \overline{DB} \) bisects \( \angle D \). \( \triangle ADB \cong \triangle CDB \) by the SSS triangle congruence condition. This makes \( \angle ADB \cong \angle CDB \), and so \( \overline{DB} \) bisects \( \angle D \).

4. Students could justify this in two ways.

**Approach 1:** Any point on the perpendicular bisector of a segment is the same distance from the endpoints of the segment. Since \( B \) is on \( EB \), the perpendicular bisector of \( \overline{AC} \), \( AB = BC \).

**Approach 2:** Since \( EB \) is the perpendicular bisector of \( \overline{AC} \), \( \angle BEA \cong \angle BEC \) and \( \overline{AE} \cong \overline{CE} \). Thus, \( \triangle BEA \cong \triangle BEC \) by the SAS triangle congruence condition. Therefore, \( \overline{AB} \cong \overline{CB} \).
1. The two triangles drawn below are not drawn to scale. In each case, determine if the given information is enough to ensure that the triangles are congruent. If they are congruent, write the congruence relation and explain your reasoning. If the given conditions cannot allow you to determine that they are congruent, explain why not.

![Diagram of two triangles](image)

a. \( \angle A \cong \angle D \)
\( \angle B \cong \angle E \)
\( \angle C \cong \angle F \)

b. \( \angle C \cong \angle F \)
\( \angle D \cong \angle A \)
\( \overline{CA} \cong \overline{FD} \)

c. \( m\angle B = m\angle E = 90^\circ \)
\( BC = DF \)
\( AB = DE \)

2. James has two pieces of spaghetti. They have lengths 5 inches and 13 inches.

   a. Is it possible to make a triangle using those two pieces and one that is 7 inches long?
b. If James wants to form a right triangle, what length should he make the third piece? Explain your reasoning.

3. Triangle $PQR$ is an isosceles triangle with $PQ = RQ$. If $m\angle P = 70^\circ$, find the measures of $\angle Q$ and $\angle R$. Explain your reasoning and show your work.

$$m\angle Q = \phantom{000}\phantom{00} \\m\angle R = \phantom{000}\phantom{00}$$

4. Sarah bought the bookcase shown below.

a. Why is it important that the diagonal bars are included in the design of the bookcase?

b. Should the diagonal bars be the same length? Explain your reasoning.
1. a. There is not enough information to determine that the two triangles are congruent. AAA does not determine exactly one triangle.
   b. $\triangle ACB \cong \triangle DFE$ by the ASA congruence condition.
   c. The triangles are not congruent. In one triangle, the pair of congruent sides form the right angle, and in the other triangle they do not.

2. a. No, it is not possible to make a triangle. The sum of the two shortest pieces of spaghetti is only 12 inches, which is less than the length of the third piece of spaghetti. Thus, it is not possible to make a triangle.
   b. There are two possible lengths for the third piece of spaghetti. If the piece of length 13 inches is the hypotenuse, then the third piece would need to have length 12 inches ($5^2 + 12^2 = 13^2$). If the third piece was the hypotenuse, then it would need to have length $\sqrt{194}$, or approximately 13.93 inches ($5^2 + 13^2 = 194$).

3. $m\angle Q = 40^\circ$; $m\angle R = 70^\circ$

4. a. The diagonal bars make the bookcase rigid and rectangular. They keep the upright sections from tilting to the left or right.
   b. Yes, the diagonal bars should be the same length. If the diagonals are not the same length, the quadrilateral determined by the two diagonals will not be a rectangle, and the shelves would not be parallel to the floor.
Think About This Situation

As you examine the photos on the previous page, try to identify the polygon in each case and think about some of its features.

a How would you describe the shape of each polygon?

b What features do each of these polygons appear to have in common?

c The design of most bolts and nuts are based on polygons with an even number of sides. Why do you think this is the case? Why do you think the nuts on many public water mains and fire hydrants have the shape shown?

d Why do you think a stop sign has the shape it has? Would a square or rectangle work just as well?

e Why do you think the cells of a honeycomb are shaped as they are? Would other polygons work just as well?

f Based on your previous work with triangles and quadrilaterals, what are some natural questions you might ask about other polygons?
## Polygons

<table>
<thead>
<tr>
<th>Number of Sides</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>Triangle</td>
</tr>
<tr>
<td>4</td>
<td>Quadrilateral</td>
</tr>
<tr>
<td>5</td>
<td>Pentagon</td>
</tr>
<tr>
<td>6</td>
<td>Hexagon</td>
</tr>
<tr>
<td>7</td>
<td>Septagon</td>
</tr>
<tr>
<td>8</td>
<td>Octagon</td>
</tr>
<tr>
<td>9</td>
<td>Nonagon</td>
</tr>
<tr>
<td>10</td>
<td>Decagon</td>
</tr>
<tr>
<td>11</td>
<td>11-gon</td>
</tr>
<tr>
<td>12</td>
<td>Dodecagon</td>
</tr>
<tr>
<td>13</td>
<td>15-gon</td>
</tr>
<tr>
<td>n</td>
<td>n-gon</td>
</tr>
</tbody>
</table>

### Some Regular Polygons

- ![Regular Octagon](image)
- ![Regular Pentagon](image)
- ![Regular Triangle](image)
- ![Regular Circle](image)
- ![Regular Hexagon](image)
- ![Regular Septagon](image)
- ![Regular Octagon](image)
- ![Regular Nonagon](image)
- ![Regular Decagon](image)
- ![Regular 11-gon](image)
- ![Regular Dodecagon](image)
- ![Regular 15-gon](image)
- ![Regular n-gon](image)
Line or Mirror Symmetry

Problems 4 and 5
Symmetry of Figures

Problem 6

A

B

C

D

E

F
In this investigation, you learned how to draw regular polygons and discovered special patterns relating the number of sides to the measure of a central angle and to the number and nature of rotational and reflection symmetries.

a Explain how you would accurately draw a regular \( n \)-gon.

b Explain how you can test to see if a figure has line symmetry. Describe the number and positions of the lines of symmetry of a regular \( n \)-gon.

c Explain how you can test if a figure has rotational symmetry. Describe the number of rotational symmetries of a regular \( n \)-gon and their angles.

Be prepared to share your ideas and thinking with the class.
Summarize the Mathematics

In this investigation, you used reasoning with triangles to help discover a pattern relating the number of sides of a polygon to the sum of the measures of its interior angles. You also discovered a surprising pattern involving the sum of the measures of the exterior angles of a polygon.

a  Explain how you can find the sum of the measures of the interior angles of a polygon. What is the measure of one interior angle of a regular $n$-gon?

b  What is true about the sum of the measures of the exterior angles of a polygon? What is the measure of one exterior angle of a regular $n$-gon?

Be prepared to share your ideas and thinking with the class.
Equilateral Triangle and Square Tessellations

Problem 1, Part c
Tiling from the Taj Mahal

Problem 5
Regular Polygons for Making Tiles

Problem 6
Summarize the Mathematics

In this investigation, you explored special polygons that tile the plane. You also investigated how combinations of those special polygons can lead to more complex patterns in the plane.

\(a\) Write a summarizing statement describing which triangles and which quadrilaterals tile the plane.

\(b\) Which regular polygons tile the plane? How do you know there are no others? Explain your reasoning.

\(c\) How do semiregular tessellations differ from regular tessellations? How can number codes be used to describe semiregular tessellations?

*Be prepared to discuss your ideas with the class.*
Nonregular Convex Pentagon Tessellations

Check Your Understanding
1. If we ignore the writing on the sign, the route marker below has one vertical line of symmetry. The sign also has the following measurements: \( m\angle A = 100^\circ, m\angle C = 110^\circ, AE = BC = 15 \text{ inches}, \) and \( AB = 21 \text{ inches}. \)

a. Is the figure a regular pentagon? Why or why not?

b. Using the symmetry of the figure and the provided measurements, find the measure of \( \angle E \) and \( \angle D \). Explain your reasoning or show your work.

\[
m\angle E = \_\_\_\_\_\_\_\_\_\_
\quad m\angle D = \_\_\_\_\_\_\_\_\_\_
\]

c. Find the perimeter of the sign.

d. Will the pentagon tile the plane?
2. a. Can a regular octagon be used to tessellate a plane? Why or why not?

b. Find the measure of the central angle of a regular octagon. Explain your reasoning or show your work.

3. The octagon in the center of the figure at the right is a regular octagon, and all of the triangles are equilateral triangles.

   a. Draw all symmetry lines for this figure.

   b. Identify the angles for all rotational symmetries of this figure. Explain your reasoning.

   c. Find the measure of $\angle ABC$ as marked. Show your work.
1. a. The figure is not a regular pentagon because not all the sides have the same length and not all the angles have the same measure.

   b. \( m \angle E = 100^\circ \) by symmetry. Also by symmetry, \( m \angle D = m \angle B \). Since the figure is a pentagon, \( m \angle A + m \angle B + m \angle C + m \angle D + m \angle E = 540^\circ \). Thus, \( 100^\circ + m \angle B + 110^\circ + m \angle D + 100^\circ = 540^\circ \). So, \( m \angle B + m \angle D = 540^\circ - 310^\circ = 230^\circ \). Since \( m \angle B = m \angle D \), you can conclude that \( m \angle D = 115^\circ \).

c. The perimeter of the sign is \( 15(3) + 21(2) = 87 \) inches.

d. This shape cannot tessellate the plane since no combination of the angles sum to \( 360^\circ \).

2. a. No, a regular octagon cannot be used to tessellate a plane. Congruent copies of an octagon cannot be placed at a vertex without overlapping because the measure of each interior angle is \( 135^\circ \), and \( 135^\circ \) is not a factor of \( 360^\circ \). Alternatively, students may indicate that 3 or more copies of this angle sum to more than \( 360^\circ \).

   b. The measure of a central angle is \( \frac{360^\circ}{8} = 45^\circ \) since there would be 8 triangles formed around the center of the octagon as shown.

3. a. This figure has four lines of symmetry.

   b. \( 90^\circ, 180^\circ, \) and \( 270^\circ \) are the angles of rotational symmetry since the figure can be rotated about the center of the octagon so that point \( A \) coincides with the outermost vertex of each of the other three triangles.

c. \( 165^\circ \). The measure of the interior angle of the octagon is \( \frac{6(180)^\circ}{8} = 135^\circ \), and the measure of the angle of the triangle is \( 60^\circ \). Thus, \( m \angle ABC = 360^\circ - 135^\circ - 60^\circ = 165^\circ \).
LESSON 2 QUIZ

Form B

1. a. Using the circle with center C below, a protractor, and a compass, accurately draw a regular hexagon. Then explain the procedure you used.

   ![Circle with center C]

   b. What is the measure of each interior angle of your hexagon? Show your work.

   c. What is the measure of each exterior angle of your hexagon?

2. a. Sketch a figure that has reflection symmetry but does not have rotational symmetry.

   b. On your sketch above, draw all symmetry lines for your figure.

   c. Explain why your shape does not have rotational symmetry.
3. a. Illustrate how you could tile the plane using the triangle below.

   ![Triangle]

   b. Can any triangle tile the plane? Explain why or why not.

4. Identify the line and rotational symmetries, if any, of the pin shown below.

   ![Pin]
LESSON 2 QUIZ

Form B
Suggested Solutions

1. a. The measure of the central angle of the hexagon is 60˚, so first draw a 60˚ angle that has its vertex at the center of the circle. The points where the sides of the angle intersect the circle determine one side of the hexagon. Then use a compass to mark off points on the circle that are the same distance apart as the two points formed by the central angle and the circle. This gives you the six vertices of the hexagon. You can draw the hexagon by connecting these six points in order around the circle. Alternatively, a student may adjust the compass to the length of the radius and mark off points on the circle using that length. (Since a hexagon is made up of six equilateral triangles, the length of the radius of the circle equals the length of a side of the hexagon.)

   b. $\frac{4(180b)}{6} = 120˚$; students may recognize that an interior angle of the hexagon is twice the measure of an equilateral triangle interior angle and write $2(60˚) = 120˚$.

   c. $180˚ - 120˚ = 60˚$

2. a. Responses will vary. See student drawings.
   
   b. See student drawings.
   
   c. It does not have rotational symmetry because it is impossible to turn it less than 360˚ and have it land back on itself.

3. a. Answers may vary. As shown below on the left, half-turns about the midpoints of the sides will tile the plane. Another method involves reflecting a triangle across a side and using this new set of two triangles to tile the plane. One example of this method is shown below on the right.

   ![Diagram](image)
b. Yes, any triangle can tile the plane. As indicated in part a, you can use repeated half-turns about the midpoints of the sides. When you rotate the triangle about the midpoint of one side of the triangle, the two triangles together will form a parallelogram. Then copies of the parallelogram can be used to tile the plane. Alternatively, the triangle can be reflected across one of its sides. Then the two triangles together will form a quadrilateral or another triangle that will tile the plane. If two copies of each angle of the triangle come together at each point, then the sum of those angles is 360°. Therefore, there will be no gaps. You must also position the triangles so that the sides match.

4. This figure is based on a regular pentagon, so it has 72°, 144°, 216°, and 288° rotational symmetry. Unlike a regular pentagon, it has no line symmetry.
Think About This Situation

Suppose you made three columns by folding and taping sheets of 8.5 × 11-inch paper so that each column was 8.5 inches high with bases shaped, respectively, as equilateral triangles (pictured at the right), squares, and regular octagons. Imagine that you placed a small rectangular piece of cardboard (about 6 by 8 inches) on top of a column. Then you carefully placed a sequence of objects (like notebooks or textbooks) on the platform until the column collapsed.

a Which column shape do you think would collapse under the least weight?

b Which column shape do you think would hold the most weight before collapsing?

c Test your conjectures by conducting the experiment with your class. What happens to the maximum weight supported as the number of sides of the column base increases?

d Suppose another column is made with a regular hexagon base. Predict how the number of objects it would support before collapsing would compare to those of the three columns above. Explain your reasoning.

e Why do you think the ancient Greeks chose to use cylindrical columns?
Polyhedron Net

Problem 2, Part a, i
Polyhedron Net

Problem 2, Part a, ii
Polyhedron Net

Problem 2, Part a, iii
Polyhedron Net

Problem 2, Part c, i
Polyhedron Net

Problem 2, Part c, ii
Polyhedron Net

Problem 2, Part c, iii
The Number of Vertices, Faces, and Edges of Polyhedra

Problem 6

<table>
<thead>
<tr>
<th>Polyhedron</th>
<th>Number of Vertices</th>
<th>Number of Faces</th>
<th>Number of Edges</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cube</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Triangular Prism</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hexagonal Prism</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Triangular Pyramid</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Square Pyramid</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
In this investigation, you explored characteristics of commonly occurring three-dimensional shapes—prisms, pyramids, cylinders, and cones.

a In what ways are polyhedra like polygons? In what ways are they different?

b What is the least number of faces that can meet at a vertex of any polyhedron? What is the least number of edges that can meet at a vertex of any polyhedron?

c How are pyramids like prisms? How are they different?

d Consider a sequence of prisms in which each base is a regular polygon. The base of the first prism has 3 sides, the base of the second has 4 sides, the base of the third has 5 sides, and so on. As the number of sides in the base increases, what shape does the prism begin to resemble?

e Consider a sequence of pyramids with bases like those described in Part d. As the number of sides in the base of a pyramid increases, what shape does the pyramid begin to resemble?

f What formula relates the number of vertices, faces, and edges of the polyhedra that you explored in this investigation?

*Be prepared to share your ideas and formula with the class.*
Square Dot Paper
A three-dimensional shape can be represented in two dimensions in various ways, including an orthographic (face-views) drawing or an oblique sketch from a particular point of view.

**a** When is it helpful to represent a three-dimensional shape by an orthographic drawing? By an oblique sketch?

**b** Discuss the similarities and differences between a top-front-right corner sketch of a right rectangular prism and the rectangular prism itself.

**c** Consider a convex polyhedron that is made up of two square pyramids sharing a common base. Make an orthographic drawing of this polyhedron. Assume an edge of the common base is parallel to an edge of your desk.

*Be prepared to share your ideas and drawing with the class.*
Summarize the Mathematics

In this investigation, you explored two types of symmetry in three dimensions: reflection symmetry and rotational symmetry. You examined the rigidity of different polyhedra. You also discovered a property about the sum of the angle defects of a polyhedron.

a Describe how to identify reflection symmetry and rotational symmetry in a polyhedron.

b Name the rigid polyhedron with the fewest faces and edges.

c What methods can be used to make a polyhedron rigid?

d What is true of the sum of the vertex angle defects for the polyhedra that you studied in this investigation?

Be prepared to share your ideas with the entire class.
Partial Polyhedron Net

Problem 5
Partial Polyhedron Net

Problem 5
Partial Polyhedron Net

Problem 5
Icosahedron Net

Problem 5
In this investigation, you demonstrated that there are exactly 5 differently shaped regular polyhedra and examined some of their symmetries.

**a** Name all the regular polyhedra according to the number of faces of each. For each regular polyhedron, describe a face and give the number of faces that meet at each vertex.

**b** Explain why there cannot be more than 5 differently shaped regular polyhedra.

*Be prepared to share your descriptions and explanation with the entire class.*
LESSON 3 QUIZ

Form A

1. a. Sketch a prism and a pyramid, showing the hidden edges, each of which has a base that is an equilateral triangle. The lateral faces of the pyramid should not be equilateral.

   Prism:  
   Pyramid:  

   b. Describe the location of all symmetry planes for the prism you drew in Part a.

   c. Describe the rotational symmetry of the pyramid you drew in Part a. Where is the axis of symmetry? What are the angles of rotation?
2. The following are three views of a cube model of a building. The height of each cube represents one story.

![Top View](image1.png)  ![Front View](image2.png)  ![Right-Side View](image3.png)

a. How tall (in stories) is the cube building? Explain how you can tell.
   
   **Height in stories:** _______________
   
   **Explanation:**

b. Describe the location of the tallest part of the cube building. Explain how you arrived at your answer.
   
   **Description and explanation:**

3. a. What is the name of the regular polyhedron shown at the right?
   
   **Name:** _______________

b. Verify that the polyhedron in Part a satisfies Euler’s Formula for Polyhedra.

c. Find the angle defect for a vertex of the regular polyhedron in Part a.
UNIT 6  Patterns in Shape

LESSON 3 QUIZ

Form A
Suggested Solutions

1. a. The triangular prism has four symmetry planes. Three of them are perpendicular to the bases and contain one lateral edge and the midpoint of the opposite side of the triangular base. The fourth symmetry plane is parallel to the bases and halfway between them.

c. The axis of symmetry is the line through the center of the base of the pyramid and the apex of the pyramid. The angles of rotation associated with this axis are 120˚ and 240˚.

2. a. The building is three stories high. The front view and the right-side view both have a part that is three blocks tall. This indicates that the building is three stories tall.

b. The tallest part of the building is the back-left corner. The tallest part is on the left of the building since the column with three blocks is on the left of the front view. You can tell it is in the back by looking at the right-side view.

3. a. regular octahedron

b. There are 6 vertices, 8 faces, and 12 edges in this polyhedron. Euler’s Formula for Polyhedra says that the number of vertices plus the number of faces equals the number of edges plus 2. Since 6 + 8 = 12 + 2, this polyhedron satisfies Euler’s Formula.

c. Four equilateral triangles meet at each of the vertices of the octahedron. Thus, the angle defect at each vertex is 360˚ – 4(60˚) = 120˚.
Form B

1. a. Is this a sketch of a right prism? Explain your reasoning.
   Yes ___________ No ___________
   Explanation:

   ![Image of a prism]

   b. Is the polyhedron shown above convex? Explain your reasoning.
   Yes ___________ No ___________
   Explanation:

   ![Image of a polyhedron]

   c. If each base is a regular hexagon, what must be true in order for the line shown at the right to be an axis of rotation?

   ![Image of a hexagon]

   d. Identify the angles of rotation associated with the axis of rotation shown in the diagram above.

   ![Image of rotation angles]

   e. Verify that Euler’s Formula for Polyhedra is satisfied by this polyhedron.
2. a. Sketch a pyramid that has a square base, showing the hidden edges.

   b. Draw a net for the pyramid you sketched.

3. At the right is a drawing of a model hotel. Assume any cube above the bottom layer rests on another cube and that there are no cubes to the right of the two visible rows.

   a. Draw top, front, and right-side views of this model hotel.

   b. Does this model hotel have reflection symmetry? If so, describe or draw the symmetry plane. If not, explain why not.

4. Explain why it is impossible to have a polyhedron that has 6 equilateral triangles meeting at a vertex.
LESSON 3 QUIZ

Suggested Solutions

1. a. Yes, it appears to be a right prism. The bases are congruent polygons that are parallel to each other, and all other sides are rectangles.

   b. It is a convex polyhedron. It is convex because any segment joining two vertices will lie either on or inside of the polyhedron.

   c. It is an axis of rotation if the line contains the center of each base.

   d. The angles of rotation associated with this axis of rotation are 60°, 120°, 180°, 240°, and 300°.

   e. There are 12 vertices, 8 faces, and 18 edges in this polyhedron. Euler’s Formula for Polyhedra says that the number of vertices plus the number of faces equals the number of edges plus 2. Since $12 + 8 = 18 + 2$, this polyhedron satisfies Euler’s Formula.

2. a. 

   b. 

3. a. Top View  

   Front View  

   Right-Side View
b. Yes, it has reflection symmetry. The symmetry plane would divide the hotel into two equal parts and would pass through the middle of the front and back of the hotel.

4. If six equilateral triangles meet at a vertex, then the sum of the angles meeting at that vertex would be 360°. But in this case, the object would not be able to be folded to make a three-dimensional shape because the angle defect would be 0°.
Shape is a fundamental feature of the world in which you live. Understanding shape involves being able to identify and describe shapes, visualize and represent shapes with drawings, and analyze and apply properties of shapes.

a Triangles and quadrilaterals are special classes of shapes called polygons.
   i. What properties are true of every polygon?
   ii. What properties are true of every quadrilateral? What property of some quadrilaterals makes the shape widely useful as a linkage?
   iii. What properties are true of every triangle?

b What does it mean for two polygons to be congruent?
   i. What information is sufficient to test whether two triangles are congruent?
   ii. Which test in part i could be used to test whether two parallelograms are congruent?
   iii. How can you use the idea of triangle congruence to reason about properties of polygons, and parallelograms in particular? What are some of these properties?

c If a statement is true, its converse may or may not be true. What is the converse of the Pythagorean Theorem? Explain why it is true. How is it used in applications?
Polyhedra are three-dimensional counterparts of polygons.

i. Compare and contrast polygons and polyhedra.

ii. Describe a variety of ways that you can represent polyhedra.

iii. In some cases, congruent copies of a polygon can be used to tile a plane. In other cases, they can be used to form a polyhedron. What must be true about the angle measures at a common vertex in each case?

iv. Compare tests for symmetries of polygons and other two-dimensional shapes with tests for symmetries of polyhedra and other three-dimensional shapes.

v. Rigidity is often an important consideration in the design of both two-dimensional and three-dimensional shapes. What is the key idea to bracing shapes for rigidity? Why does this work?

Be prepared to share your descriptions and reasoning with the class.
UNIT 6  Patterns in Shape

Name  ____________________________
Date  _____________________________

UNIT SUMMARY

In this unit, you studied some of the geometry of two- and three-dimensional shapes. You learned how to identify and classify shapes, how to visualize and represent them, and how to analyze and apply some of their properties. You learned why some of these shapes are used so frequently in building and design. In addition, you began to develop the ability to carefully reason from definitions and known or assumed facts to new facts.

State the Triangle Inequality and explain why it is useful.

Describe the relationship that must be satisfied by the four side lengths of any quadrilateral.

Describe characteristics of a quadrilateral linkage that make it useful in the design of mechanical devices.

Suppose \( \triangle ABC \cong \triangle PQR \). Draw a diagram that illustrates this situation.

- Use markings on your diagram to indicate corresponding sides and corresponding angles that are congruent.
- Write 6 congruence statements that must be true.

Describe sets of side and angle conditions that can be used to test whether two triangles are congruent.

- 
- 
- 
-
Describe how the Triangle Angle Sum Property can be used to reason to the corresponding property for:
- any quadrilateral. State the property.__________________________________________________________
- any polygon. State the property.____________________________________________________________

The Pythagorean Theorem is one of the most important theorems in geometry. State the theorem and its converse and explain when and how to use both cases.

Give a mathematical example of a true statement where the converse statement is not true.

Use the graphic organizer below to indicate how the special quadrilaterals (square, rhombus, parallelogram, rectangle, and kite) are related to each other.

![Graphic Organizer]

Draw a rectangle and its diagonals. Explain how you could justify that the two diagonals are congruent by:
- using careful reasoning and the Pythagorean Theorem.
- using careful reasoning and congruent triangles.
Describe how you can identify each of the following types of symmetry for two-dimensional figures. Then draw an example of a figure that has each type of symmetry.

- Reflection symmetry
- Rotational symmetry
- Translation symmetry

Only three regular polygons tile the plane. Identify those polygons and explain why they and no others will tile the plane.

Describe how you can make a polygon or a polyhedron rigid.

Polyhedra are special three-dimensional shapes formed by polygons.

- Describe similarities and differences between prisms and pyramids. Draw a sketch of each.
• Select a polyhedron and make an orthographic (face-view) drawing and a sketch from a particular point of view that shows any hidden edges.

• Sketch an example of a polyhedron that has plane symmetry and rotational symmetry. Describe the symmetries.

• State Euler’s Formula for polyhedra.

• Name and describe the five regular polyhedra.

Summarize what you know about the following angle relationships.

• The sum of the measures of the interior angles of a polygon with $n$ sides: _______________________

• The sum of the measures of the exterior angles of a convex polygon: _______________________

• The measure of an interior angle of a regular polygon: _______________________

• The measure of an exterior angle of a regular polygon: _______________________

• The measure of a central angle of a regular polygon: _______________________

• The angle defect at a vertex of a convex polyhedron: _______________________

• The sum of the angle defects of a convex polyhedron: _______________________


1. The polyhedron below is formed by taking a square prism and placing a square pyramid on top of it. The triangles that form the pyramid on top of the prism are all equilateral triangles.

![Polyhedron Diagram]

a. Identify all reflection symmetries or rotational symmetries for the polyhedron. If the figure has reflection symmetry, draw or describe the plane(s) of reflection. If it has rotational symmetry, describe any axes of symmetry and the associated angle(s) of rotation.

Reflection symmetries:

Rotational symmetries:

b. Find the angle defect for one of the vertices where the prism and the pyramid meet. Show your work.

Angle defect: ____________________

c. Is this polyhedron convex? Explain your answer.
2. In the isosceles trapezoid below, $AB = BC = CD = 5$ centimeters.

![Isosceles trapezoid diagram]

a. Identify all reflection symmetries and rotational symmetries for the trapezoid. If it has reflection symmetry, draw or describe the line(s) of reflection. If it has rotational symmetry, describe the angles(s) of rotation.

*Reflection symmetries:*

*Rotational symmetries:*

b. Describe all possible lengths for $\overline{AD}$.

c. If the measure of $\angle A$ is $30^\circ$, find the measures of all the remaining angles. Explain how you determined the angle measures.

$m\angle B = \underline{\hspace{2cm}} \quad m\angle C = \underline{\hspace{2cm}} \quad m\angle D = \underline{\hspace{2cm}}$

d. If the length of the fourth side of the trapezoid is 11 centimeters, is the height of the trapezoid 4 centimeters? Explain why or why not.
3. Consider the regular 12-sided polygon that is drawn below. The center of the regular polygon is point $O$.

![Regular 12-sided polygon with center $O$]

a. Find the measure of each interior angle of the polygon. Show your work below.

$Measure$ $of$ $each$ $interior$ $angle$: __________

b. What is the measure of an exterior angle of the polygon? Explain how you got your answer.

$Measure$ $of$ $an$ $exterior$ $angle$: __________

c. Can regular 12-sided polygons and equilateral triangles be used to make a semi-regular tessellation of the plane? Explain your reasoning.

d. Does this polygon have 160˚ rotational symmetry? Explain your reasoning.
4. a. Each person in your class has drawn a triangle that has sides of length 10, 15, and 18 centimeters. Must all of your triangles be congruent? Explain why or why not.

   b. Each person in your class has drawn a parallelogram that has two sides of length 8 centimeters and two sides of length 12 centimeters. Must all of your parallelograms be congruent? Explain why or why not.

5. In the truss below, the indicated segments are all the same length. Additionally, \( \angle ACD \equiv \angle BCE \).

   a. Explain why \( AD \equiv BE \).

   b. Explain why \( \angle BAC \equiv \angle ABC \).
UNIT TEST

Form A

Suggested Solutions

1. a. There are four symmetry planes for this polyhedron. They correspond to the four lines of symmetry for the square. They are all vertical planes. Two of them contain the midpoints of opposite sides of the square, and the other two contain opposite vertices of the square.

There is only one axis of rotation. It contains the apex of the pyramid and is perpendicular to the square that the polyhedron is resting on. The angles of rotation associated with this axis of symmetry are 90°, 180°, and 270°.

b. Two rectangles and two triangles meet at each of the indicated vertices. Thus, the angle defect is $360° - (90° + 90° + 60° + 60°) = 60°$.

c. Yes, the polyhedron is convex because any segment joining two vertices will be on or inside the polyhedron.

2. a. The isosceles trapezoid has one line of reflection. It is a vertical line passing through the midpoints of the two bases of the trapezoid. The trapezoid has no rotational symmetry.

b. The length of the fourth side must be less than 15 centimeters, because the sum of the lengths of the other three sides must be greater than the length of the fourth side.

c. $m\angle D = 30°$ by symmetry. Since the sum of the interior angles of a quadrilateral must be $360°$, $m\angle B + m\angle C = 300°$. By symmetry, $\angle C$ and $\angle B$ must have equal measures.

So, $m\angle C = m\angle B = 150°$.

d. Yes. Draw line segments from $B$ and $C$ that are perpendicular to the longer base of the trapezoid. A rectangle is formed, so the middle segment of the base is 5 cm. Thus, each end segment must have length 3 cm.

![Diagram of trapezoid]

Since the triangles formed are right triangles, you can use the Pythagorean Theorem to determine that the height of the trapezoid must be 4 cm.

3. a. The measure of each interior angle is $\frac{(12-2)180}{12} = 150°$. 
b. The measure of each exterior angle is 30˚. Students can either find this by evaluating \( \frac{360\text{b}}{12} \) to get 30˚, or by realizing that the measures of the exterior angle and the interior angle must sum to 180˚, so the exterior angle must have measure 30˚.

c. Yes, equilateral triangles and regular 12-sided polygons can be used to tile the plane as shown below.

![Diagram of equilateral triangles and 12-sided polygons tiling the plane]

The interior angles of the 12-sided polygons are each 150˚, and two of these angles come together at each vertex. Also, the interior angles of the equilateral triangle measure 60˚, and one of those meets the two 12-sided polygons at each vertex. So, 150˚ + 150˚ + 60˚ = 360˚. This pattern can be continued to tile the plane. Thus, there are no gaps.

d. No, the polygon does not have 160˚ rotational symmetry. The measure of a central angle is 30˚ so rotational symmetries of the polygon must be multiples of 30˚, and 160˚ is not.

4. a. Yes, all of the triangles will be congruent. The lengths of three sides of a triangle determine the triangle. This is the SSS triangle congruence condition.

b. No, all of the parallelograms will not necessarily be congruent. In fact, one of them could be a rectangle and others could be parallelograms that are not rectangles, as shown below.

![Diagram of a rectangle and a parallelogram]

5. a. \( \triangle ACD \cong \triangle BCE \) by the SAS triangle congruence condition. Thus, \( \overline{AD} \cong \overline{BE} \) because they are corresponding parts.

b. \( \angle BAC \cong \angle ABC \) because \( \triangle ABC \) is isosceles and the angles opposite the congruent sides are congruent.
1. Suppose that you have 4 straws of length 6 cm and 2 straws of length 8 cm.
   a. How would you use those 6 straws to make a hexagon that has exactly two lines of symmetry? Draw a sketch of your hexagon below. Label the length of each side of your hexagon. Also draw the two lines of symmetry on your hexagon.
   
   b. Describe all rotational symmetries for your hexagon.
   
   c. Kai used those same straws to make a hexagon with exactly two lines of symmetry. Must her hexagon be congruent to yours from Part a? Explain why or why not.
   
   d. Would it be possible to make a parallelogram using any four of these straws so that the parallelogram has a diagonal with length 14 centimeters? Explain why or why not.
   
   e. Ron used the two straws that are 8 cm long and two of the 6-cm-long straws to make a parallelogram. The parallelogram has a diagonal of length 10 cm. Is the parallelogram a rectangle? Explain why or why not.
2. Polygon \(ABCD\) below is a regular pentagon.

![Pentagon Diagram]

a. Find the measure of \(\angle A\).

b. Identify two congruent triangles in the figure above. How do you know that they are congruent?

c. Could pentagon \(ABCD\) be used to tile the plane? Explain why or why not.

3. The gambrel truss shown below is used when making barn-shaped roofs. The boards \(AH, HC, LE,\) and \(LG\) are all the same length. Additionally, bottom board \(AG\) is divided into six congruent sections.

![Gambrel Truss Diagram]

a. Explain why \(\angle A \cong \angle HCB\).

b. Explain why board \(LF\) must be perpendicular to bottom board \(EG\).
4. The net below is for a polyhedron.

![Net of a polyhedron](image)

a. Name the polyhedron for which this is a net.

b. Make a three-dimensional drawing of the polyhedron, showing hidden edges.

5. The figure below is a rectangular pyramid.

![Rectangular pyramid](image)

a. Describe all reflection symmetries for this pyramid.

b. Describe all rotational symmetries for this pyramid, including the angles of rotation.

6. Pick either Euler’s Formula for Polyhedra or Descartes’ Theorem and verify that it is satisfied by a regular octahedron.
UNIT TEST

Form B

Suggested Solutions

1. a. For selected students, you may wish to provide straws to assist them in completing this task.

![Hexagon Diagram](image)

b. The above hexagon has 180° rotational symmetry about the center of the hexagon.

c. The hexagons will not necessarily be congruent. Although the sides have the same measures, the angles do not need to be the same. The hexagon shown below has exactly two lines of symmetry but is not congruent to the one shown above.

![Second Hexagon Diagram](image)

d. It is not possible to make a parallelogram with a 14-cm long diagonal from these straws. Since a parallelogram must have two pairs of congruent sides, this parallelogram might have two sides of length 6 cm and 2 sides of length 8 cm. But then by the Triangle Inequality, the diagonal of the parallelogram must be less than $6 + 8 = 14$ cm. If the parallelogram (rhombus) had four sides of length 6 cm, the diagonal would need to be less than 12 cm.

e. Yes, the parallelogram is a rectangle. Since $6^2 + 8^2 = 10^2$, we know by the converse of the Pythagorean Theorem that the triangle formed by the sides of the parallelogram and the diagonal must be a right triangle. But a parallelogram with one right angle must be a rectangle.

2. a. \[ \angle A = \frac{3(180 \text{°})}{5} = 108 \text{°} \]

b. \( \triangle ABE \cong \triangle CBD \) by the SAS congruence condition. We know that \( \overline{AB} \cong \overline{CB}, \overline{EA} \cong \overline{DC}, \) and \( \angle A \cong \angle C \) since all sides of a regular pentagon are congruent and all angles are congruent.
c. No, pentagon $ABCDE$ cannot be used to tile the plane since $108^\circ$ is not a factor of $360^\circ$.

3. a. Since $\overline{AH} \cong \overline{HC}$, $\triangle AHC$ is an isosceles triangle. Thus, the angles opposite the congruent sides will be congruent. So, $\angle A \cong \angle HCA$. Alternatively, students could say that $\triangle ABH \cong \triangle CBH$ by the SSS triangle congruence condition. Then since $\angle A$ and $\angle HCA$ are corresponding parts of these two triangles, they must be congruent.

b. $\triangle ELF \cong \triangle GLF$ by the SSS triangle congruence condition. Thus, we know that $\angle LFE \cong \angle LFG$. But additionally, we know that $m \angle LFE + m \angle LFG = 180^\circ$. So, since they are congruent and their measures sum to $180^\circ$, each must be $90^\circ$. So, $\overline{LF} \perp \overline{EG}$.

4. a. Tetrahedron

b. 

5. a. The pyramid has two planes of reflection. They each contain the apex of the pyramid and go through the midpoints of two opposite sides of the rectangular base.

b. The pyramid has only one axis of rotation. It contains the apex of the pyramid and the center of the base. The angle of rotation associated with this axis of symmetry is $180^\circ$.

6. To verify Euler’s Formula for Polyhedra, you need to show that the number of vertices plus the number of faces equals the number of edges plus 2. An octahedron has 6 vertices, 8 faces, and 12 edges. Since $6 + 8 = 12 + 2$, Euler’s Formula is satisfied.

To verify Descartes’ Theorem, you need to show that the sum of the angle defects for the octahedron is $720^\circ$. The angle defect for each vertex is $360^\circ - 4(60^\circ) = 120^\circ$. There are six vertices, and since $6(120^\circ) = 720^\circ$, Descartes’ Theorem is satisfied.
1. Consider the sequence of shapes shown below.

![Shapes](image_url)

a. Describe what shape $n$ would look like.

b. Identify and describe all the reflection symmetries for shape $n$.

c. Identify and describe all the rotational symmetries for shape $n$.

2. In this unit, you developed the formula for finding the sum of the interior angles of any convex polygon based on the number of sides of the polygon. The formula that you developed is $S = 180°(n - 2)$.

a. Explain how you arrived at this formula.

b. Will and Jessica were trying to develop this formula and drew the diagrams below.

![Diagrams](image_url)

They then reasoned as follows.

*We know that the measures of the angles of every triangle will add up to 180°. Since I can divide a polygon with $n$ sides into $n$ triangles, the sum of the interior angles of the polygon must be $180n$.*

Is the formula that Will and Jessica developed equivalent to the one that you developed during this unit? Explain how you know.

c. Use the diagrams that Will and Jessica drew to develop a correct formula for finding the sum of the interior angles of a convex polygon. Explain your reasoning.
3. You or someone you know may make quilts. But even if you don’t know anyone who makes quilts, you have probably seen some beautiful handmade quilts. The squares that are put together to make a quilt often contain geometric designs. In this task, you will have the opportunity to explore and analyze some of the designs that can be used to make quilts.

a. Using the Internet or books about quilting, find a design for a quilt square that fits each of the following criteria. You should either draw a sketch of each quilt square, make a copy of each, or print each one off the Internet. Then describe how it satisfies the given criteria.

   i. Has both rotational and reflection symmetry
   ii. Has rotational symmetry but does not have reflection symmetry
   iii. Has rotational symmetry other than 90°, 180°, or 270° rotational symmetry
   iv. Has reflection symmetry but does not have rotational symmetry
   v. Has no rotational or reflection symmetry

b. Choose one of your quilt-square patterns from Part a and use either paper or cloth to actually make one square. Your finished square should be 8 inches on a side. Then describe each of the pieces that you used in making your quilt square.
1. a. Shape $n$ will have $n + 2$ equally spaced congruent branches coming out of the center.

b. Each shape has $n + 2$ lines of reflection. If $n$ is even, there will be $\frac{n + 2}{2}$ lines of reflection that bisect a pair of branches, and there will be $\frac{n + 2}{2}$ lines of reflection that contain the center of the shape and bisect the space between two branches. See Shape 4 below with its lines of reflection drawn in.

![Diagram of Shape 4 with lines of reflection]

If $n$ is odd, each line of symmetry will bisect one of the branches. Shape 3 with its lines of reflection is shown below.

![Diagram of Shape 3 with lines of reflection]

c. Shape $n$ will have $n - 1$ rotational symmetries. The degrees of rotation will be the first $n - 1$ multiples of $\frac{360^\circ}{n + 2}$.

2. a. The formula was developed by drawing all the diagonals from one vertex of the polygon. These diagonals divide the polygon into $n - 2$ triangles. Since the sum of the angles of these $n - 2$ triangles is equal to the sum of the interior angles of the polygon, you know that the sum of the angles of the polygon will be $180^\circ(n - 2)$.

b. The formula that Will and Jessica developed is not equivalent to the one developed during the unit. Students might use symbolic reasoning to show that they are not equivalent, or they might just show that the two formulas do not give the same angle sum for any one particular value of $n$. 

c. The sum of the angles of the triangles that Will and Jessica drew includes all of the angles where the triangles meet in the middle of the polygon. Since these angles tessellate the point where they meet, the sum of these angles is 360°. To correct the formula, you must subtract 360° from the $180n$. So, the correct formula is $S = 180n - 360°$.

3. There are many resources both on the Internet and in print that students can use to find different quilt patterns. Be sure that students not only find the patterns for the quilt squares but that they also describe how each square satisfies the indicated criteria. You will need to make it clear to students whether or not they should consider color when analyzing the symmetries of each pattern. It is probably easier for students to see the symmetries if they do consider color. After students complete this task you may wish to display the different quilt blocks so that students can see the variety of patterns available and how color can enhance the block.

One Web site that has many different patterns is www.quiltbus.com/QuiltBlocks.htm, but students can easily find other diagrams and/or pictures of quilt blocks. One possible drawback of the above site is that it provides directions for making each quilt block. However, not all of the patterns are for 8-inch squares.
UNIT 6  Patterns in Shape

PROJECT

Archimedean Solids

Purpose
In this unit, you explored properties of three-dimensional solids. In particular, you analyzed prisms, pyramids, and Platonic solids. In this project, you will learn about and analyze Archimedean solids. The shape of a soccer ball is one example of an Archimedean solid.

Directions
1. Begin by researching Archimedean solids. Find out how many there are and how you can determine whether or not a three-dimensional shape is an Archimedean solid.

2. Create models of three different Archimedean solids. You can make your models using nets of the solids, or you can create them using straws and pipe cleaners, or any other materials that will work for you.

3. Now explore your shapes. Be sure your exploration allows you to answer the following questions.
   • What are the names of your three shapes?
   • What polygons are used for the faces of your shapes?
   • What types of polygons meet at each vertex? Are they the same for every vertex?
   • How are your shapes related to, similar to, or different from the polyhedra that you explored in this unit?
   • Does Euler’s Formula for Polyhedra hold for your shapes?
   • Does Descartes’ Theorem hold for your shapes?

4. Prepare a paper, a poster, or a video that demonstrates what you have learned about Archimedean solids. You should be sure to include general information about Archimedean solids as well as information about the specific solids you created. Be sure to answer any questions that were asked in Parts 1–3 of this project.
Suggested Solutions
Archimedean Solids

This project could be done individually or in pairs. It will require students doing independent research on Archimedean solids. There is an abundance of information on the Internet related to Archimedean solids. Students should be able to find definitions, examples, diagrams, as well as instructions for creating the solids. Several Web sites that are particularly helpful are listed:

mathworld.wolfram.com/ArchimedeanSolid.html
www.mathconsult.ch/showroom/unipoly/background.html
www.scienceu.com/geometry/facts/solids/
users.adelphia.net/~eswab/Archimedean.html
www.uwgb.edu/dutchs/symmetry/archpol.htm

Suggested Timeline
Students should be able to find the definition in one evening. They will then probably need several days to create and analyze the solids. They will need several more days to prepare their paper, poster, or video.

Solutions
1. An Archimedean solid, also called a semi-regular solid, is a convex polyhedron where all the faces are regular polygons, and there are at least two different regular polygons used as faces. Additionally, the arrangement of shapes at each vertex is identical. There are thirteen different Archimedean solids pictured below.
2. Allow students to create the solids using any method that works for them. You may need to remind them of how you created the prisms and pyramids during the unit. If they choose to use nets to make their solids, they can find instructions for creating the solids on the Internet. Some students may have construction toys that could be used to build these solids.

3. The table below identifies the polygons that meet at each vertex and the number of vertices, edges, and faces in each of the solids. Students should be able to verify that both Euler’s Formula and Descartes’ Theorem hold for these convex solids.

<table>
<thead>
<tr>
<th>Solid</th>
<th>Polygons at Each Vertex</th>
<th>Vertices</th>
<th>Edges</th>
<th>Faces</th>
</tr>
</thead>
<tbody>
<tr>
<td>cuboctahedron</td>
<td>3,4,3,4</td>
<td>12</td>
<td>24</td>
<td>14</td>
</tr>
<tr>
<td>great rhombicosidodecahedron</td>
<td>4,6,10</td>
<td>120</td>
<td>180</td>
<td>62</td>
</tr>
<tr>
<td>(truncated icosidodecahedron)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>great rhombicuboctahedron</td>
<td>3,4,4,4</td>
<td>48</td>
<td>72</td>
<td>26</td>
</tr>
<tr>
<td>(truncated cuboctahedron)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>icosidodecahedron</td>
<td>3,5,3,5</td>
<td>30</td>
<td>60</td>
<td>32</td>
</tr>
<tr>
<td>small rhombicosidodecahedron</td>
<td>3,4,5,4</td>
<td>60</td>
<td>120</td>
<td>62</td>
</tr>
<tr>
<td>(rhombicosidodecahedron)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>small rhombicuboctahedron</td>
<td>4,6,8</td>
<td>24</td>
<td>48</td>
<td>26</td>
</tr>
<tr>
<td>(rhombicuboctahedron)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>snub cube</td>
<td>3,3,3,3,4</td>
<td>24</td>
<td>60</td>
<td>38</td>
</tr>
<tr>
<td>snub dodecahedron</td>
<td>3,3,3,3,5</td>
<td>60</td>
<td>150</td>
<td>92</td>
</tr>
<tr>
<td>truncated cube</td>
<td>3,8,8</td>
<td>24</td>
<td>36</td>
<td>14</td>
</tr>
<tr>
<td>truncated dodecahedron</td>
<td>3,10,10</td>
<td>60</td>
<td>90</td>
<td>32</td>
</tr>
<tr>
<td>truncated icosahedron</td>
<td>5,6,6</td>
<td>60</td>
<td>90</td>
<td>32</td>
</tr>
<tr>
<td>truncated octahedron</td>
<td>4,6,6</td>
<td>24</td>
<td>36</td>
<td>14</td>
</tr>
<tr>
<td>truncated tetrahedron</td>
<td>3,6,6</td>
<td>12</td>
<td>18</td>
<td>8</td>
</tr>
</tbody>
</table>

**Report Format**

Since the Archimedean solids are hard to draw and it is much easier to talk about the solids if you can see them, this project seems especially well suited to video presentation. It would be good to allow students to decide which is the best way for them to convey what they have learned about Archimedean solids. Encourage them to be creative but also to be sure that they have answered all of the questions.
Purpose
In this unit, you learned about using orthographic and oblique drawing to represent three-dimensional shapes on a piece of paper. Artists or architects who want realistic renderings of buildings or other three-dimensional objects use perspective drawings with one or more vanishing points.

Directions
1. The fact that we see the world in perspective is dramatically illustrated by looking down a flat road lined by equally spaced telephone poles. What we see is very much like the perspective sketch here. In the real world, the sides of the road are parallel, pairs of consecutive telephone poles are about the same distance apart, and telephone poles all have about the same height. Describe how the sides of the road and the telephone poles appear to you as you see them receding into the distance.

2. In a one-point perspective sketch, parallel edges that recede from the viewer are not drawn parallel, but rather they are drawn to intersect in a common vanishing point when extended. Do some research on one-point perspective drawing and then use geometry software to explore one-point perspective sketches of rectangular prisms. (There is an applet that you can use for this exploration at www.wmich.edu/cpmp/applets/1-tperspective.html.) In particular, explore what happens when you vary the location of the vanishing point. As you do so, determine answers to these questions.
   i. Describe the location of the vertices as you vary the location of the vanishing point. What lines do these vertices lie on?
   ii. Which edges are parallel for any location of the vanishing point?
   iii. Are any edges left unchanged for any location of the vanishing point? If so, which one(s)?
   iv. Which edges change in length when the vanishing point changes position? How do their lengths change with respect to the vanishing point?

3. After completing your exploration, make a one-point perspective sketch of a cube viewed from a point in front, slightly above, and slightly to the left. Identify the vanishing point on your sketch and describe how you made your sketch. Then make an oblique drawing of a cube from the same perspective. Describe the differences and similarities between your two drawings.
4. Complete one of the following:
   - Make a one-point perspective drawing of a scene. You may choose to either draw the inside of a room or an outside landscape. Clearly identify the vanishing point of your drawing.
   - Find three examples of artwork that was created using one-point perspective. (Your artwork can be advertising in a magazine, drawings in a book, or paintings or drawings done by artists.) For each piece of art, identify the vanishing point that was used to make the drawing. Explain how you identified each vanishing point.

5. Gather all of your work together and organize it so that it clearly shows what you have learned about one-point perspective drawing. Be sure that you have answered all of the questions posed in Parts 1–4.
This project allows students to explore another way of representing three-dimensional objects in two dimensions. Perspective drawing can be done using one or more vanishing points. In this project, students will explore one-point perspective drawing. It is possible that they may have previously done some of this type of drawing in art classes. If that is the case, be sure that the students focus on the geometrical properties in the original objects and those in the drawing. This project is best done individually.

**Note:** Students should use the “Exploring One-Point Perspectives” applet for this project. This applet is located at www.wmich.edu/cpmp/applets/1-ptperspective.html.

**Suggested Timeline**

Students will be able to complete initial exploration of one-point perspective drawing in one or two evenings. You may then wish to have a short discussion of it during class. Students will then need several more days to compare one-point perspective drawing and oblique drawing and to either create or find examples of one-point perspective drawings. You may wish to look at a draft of student work and offer comments for revision. Students can then revise their work and submit a final project.

**Solutions**

1. The sides of the road appear to get closer together the farther they are from our view, finally meeting at the horizon. Similarly, the telephone poles appear to get shorter and closer together as they recede into the distance.

2. i. The front face remains fixed, and vertices not on the front face lie on a line containing a vertex of the front face and the vanishing point.

   ii. Edges that are parallel in the front and back faces remain parallel. Back-to-front parallel edges are no longer parallel; if extended they would intersect in the vanishing point.

   iii. The edges of the front face are left unchanged for any location of the vanishing point.

   iv. All edges except those of the front face change in length as the vanishing point changes. As the vanishing point is moved farther away, the edges of the back face become shorter, and back-to-front edges become longer.

3. Draw a square $ABCD$ labeled clockwise with $A$ in the upper right vertex. This is the front face of the cube. Choose a vanishing point $V$ that is located to the left and above the square. Place a straightedge...
along the line joining the lower-right vertex \( C \) of the cube and \( V \). Choose a point \( G \) on line \( CV \) between \( C \) and \( V \) so that the length of segment \( CG \) is somewhat less than an edge of the front face. Try different lengths to get the effect of depth that you want. (In technical perspective drawing, this length is determined exactly.) Once you choose \( G \), use the facts that \( EFGH \) is a square with edges parallel to those of the front face \( ABCD \) and that back-to-front edges go toward the vanishing point. For example, vertex \( F \) is the point of intersection of line \( BV \) and the line through \( G \) that is parallel to front edge \( BC \). Once segment \( FG \) is determined, the square that represents the back face of the cube is easy to construct, as is the remainder of the cube.

To make an oblique drawing of a cube, start in the same way as in Part c by drawing a square \( ABCD \) labeled clockwise with \( A \) in the upper-right vertex. This is the front face of the cube. Then draw a copy of square \( ABCD \) offset from the original square, and label it \( EFGH \). This is the back face of the cube. Next, join \( A \) and \( E \), \( B \) and \( F \), and \( C \) and \( G \), and \( D \) and \( H \). These segments will be parallel to one another, rather than going toward a vanishing point as in a perspective drawing.

4. Giving students the choice of either drawing a one-point perspective sketch or finding examples of one-point perspective drawings allows them to use their own strengths. It should not be too hard for students to find examples of drawings that make use of one-point perspective. They can find examples in books of art from the Renaissance, magazine advertisements, or even comic books.

**Report Format**

You may want students to organize their work into two parts. The first will be the work that they do on Parts 1–3 of the project. This work is probably most easily submitted in a report format. Students should be sure to answer all questions and clearly convey their thoughts. The second part of this project, Part 4, may be most easily done as a poster presentation. Students can mount their own drawing or the artwork that they found on a piece of poster board that you can then display in the classroom. This allows for a portion of the project to be shared with all of the students in the class.
Solve each problem. Then record the letter that corresponds to the correct answer.

1. If $a$ is a positive integer and $a^2 = 4$, what is $(-3)^a$?
   (a) $-9$  (b) $-6$  (c) $6$  (d) $8$  (e) $9$

2. For positive integers $x$ and $y$, if $x < y$, which of the following statements is false?
   (a) $\frac{1}{x} > \frac{1}{y}$  (b) $-x > -y$  (c) $x - y < 0$  (d) $\frac{1}{x} < \frac{1}{y}$  (e) $\frac{y}{x} > 1$

3. The perimeter of a rectangle is 28. The ratio of the width and length of the rectangle is 3:4. What is the length of a diagonal of the rectangle?
   (a) 5  (b) 10  (c) 20  (d) 22.1  (e) 22.8

4. One pound is approximately 0.454 kilograms. Sue weighs the equivalent of 55 kilograms. Approximately what is her weight in pounds?
   (a) 25 lb  (b) 108 lb  (c) 110 lb  (d) 121 lb  (e) 150 lb

5. The table below shows the number of books that a sample of 25 students carry to school. What is the median number of books?

<table>
<thead>
<tr>
<th>Number of Books</th>
<th>Number of Students</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>7</td>
</tr>
<tr>
<td>6</td>
<td>5</td>
</tr>
<tr>
<td>7</td>
<td>5</td>
</tr>
</tbody>
</table>

   (a) 3  (b) 4  (c) 5  (d) 6  (e) 7

6. A number $x$ is multiplied by itself, and the result is added to 3 times the original number. This can be expressed algebraically as:
   (a) $x + 3$  (b) $x^2 + 3$  (c) $2x + 3$  (d) $x^2 + 3x$  (e) $2x + 3x$

7. The enrollment at Cedar Creek High School this year is 1,250 students. Last year the enrollment was 1,000. By what percent did the enrollment change between last year and this year?
   (a) 20%  (b) 25%  (c) 80%  (d) 125%  (e) 250%

8. Which of the following numbers is the smallest?
   (a) $3.2 \times 10^{-3}$  (b) $3.2 \times 10^1$  (c) $3.82 \times 10^{-4}$  (d) $3.82 \times 10^4$  (e) $3.82 \times 10^{-3}$
9. A shop announces a clearance sale. The price of each item is 60% off. The original price of a watch is $65. By how many dollars will the price of the watch be reduced?

(a) $26  (b) $39  (c) $46  (d) $49  (e) $56

10. Which figure has the largest area?

(a)  
(b)  
(c)  
(d)  
(e)  

**Test-Taking Tip**

Ratios and proportions can be helpful in solving problems involving percents.

Example  Look back at Item 7. The amount of enrollment change was 250 students. To find the percent change, use the ratio of change amount to the beginning value. So, the percent change is equivalent to \( \frac{250}{1,000} = \frac{25}{100} \), or 25%.

Find, if possible, another test item in the practice set for which ratios and/or proportions can be used to help find a percentage.